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Approximate Analytic Solution For Forced Korteweg-De Vries Equation With Periodic Forcing

Vincent Daniel David[1], Sharifah Wan Nurazreen, Arifah Bahar[2,3] and Zainal Abdul Aziz[2, 3]

[1]*School of Mathematical Sciences, College of Computing, Informatic and Media, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia*
[2]*Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Malaysia*
[3]*Centre for Industrial and Applied Mathematics, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor*
Correspondent author: vincent@fskm.uitm.edu.my

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Abstract: In this article, forced Korteweg-de Vries (fKdV) equation with a general periodic forcing is solved using Homotopy Analysis Method (HAM). The fKdV model used is nondimensionalized and simplified using transformation to get a general form of fKdV equation. By using a specific initial guess function, HAM solution of general form of fKdV is found at seventh order series solution. The horizontal line segment of HAM convergence series confirming the validity of the found solutions. The convergence series is shown for various *t* and it is found the canonical equation of KdV with forcing function creates disturbances over the profile. The HAM solution elucidates that the method is effective and provides a novel way to obtain convergence series solution for nonlinear problems.

Keywords: XXXXXX

INTRODUCTION

Forced KdV equation had attracted the attentions of researchers, mathematicians and scientist throughout the world for more than five decades ago [1-4]. Forced KdV equation under different parameter setting is used to explained interaction of profile with under certain forcing terms [1]. Wu and Wu found numerically a phenomenon whereby a forcing disturbance generated by moving surface pressure or topography based on Boussinesq model [3]. Ship models moving in a towing tank with various speeds had been investigated experimentally and it had been relating to fKdV equation [5]. Shen [6] had derived the third order of KdV equation with forcing term and shows its solution. Analytical model of tsunami generation by using fKdV equation was proposed by Efim Pelinovsky [7]. Forced KdV equation is nonhomogeneous, and it only can be obtained by perturbation or numerical techniques [8]. Therefore, it is important to use a powerful and reliable method to solve fKdV equation.

Liao [9] introduced an analytic approach namely homotopy analysis method (HAM) to solve various types of nonlinear equations, including ordinary differential equations, partial differential equations and differential-integral equations. HAM has a greater advantage in solving strong nonlinear models without depending on a small parameter as in the perturbation techniques. HAM consists of Lyapunov's small

Corresponding Author: Vincent Daniel David, School of Mathematical Sciences, College of Computing, Informatic and Media, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia. Email: vincent@fskm.uitm.edu.my

parameter method, the δ-expansion method and Adomian's decomposition method [10]. The validity and effectiveness of the HAM solutions had been verified through many successful applications in science, engineering and finance [11-13].

Recently, forced KdV model for some certain forcing functions had been approximated by using HAM [14-16]. Forced KdV model with periodic forcing had been attempted by several researchers [17-19]. This research extends the work from David et al [17], in which the relation of periodic forcing term in fKdV and bottom topography is investigated. The fKdV model from Grimshaw [20] is nondimensionalized and simplified using transformation to get a general form of fKdV equation. Periodic function employed onto fKdV equation and it is solved by HAM. The obtained solution able to depicts a reasonable result.

Mathematical Formulation of Forced Korteweg-de Vries

The fKdV model used in this work is [20]:

$$-\eta_t - \Delta\eta_x + \frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx} + \frac{1}{2}f_x\left(x\right) = 0 \tag{1}$$

subject to initial condition,

$$\eta\left(x,0\right) = g\left(x\right) \tag{2}$$

where η is water wave elevation, f is forcing function and Δ is a critical parameter.

The equation (1) is non-dimensionalized using the following parameter.

$$t^* = \frac{1}{6}t,\ \eta^* = \frac{3}{2}\eta,\ f^* = \frac{9}{2}f,\ \text{and}\ \Delta^* = 6\Delta\ . \tag{3}$$

Using (3), equation (1) rewritten with superscript is omitted,

$$\eta_t + \Delta\eta_x - 6\eta\eta_x - \eta_{xxx} - f_x\left(x\right) = 0 \tag{4}$$

Then, the forcing term, f will be representing underwater bottom geometry. It is easier to pass over the system of coordinates with the underwater moving obstacle. Let the forcing term, f corresponds to a moving frame of reference.

$$x^* = x - Vt\ \text{ and }\ t^* = t \tag{5}$$

From (5),

$$\frac{\partial}{\partial t} = -V\frac{\partial}{\partial x^*} + \frac{\partial}{\partial t^*}\ \text{ and }\ \frac{\partial}{\partial x} = \frac{\partial}{\partial x^*} \tag{6}$$

Using (6), the equation (4) is re-written.

$$\left(-V\frac{\partial\eta}{\partial x^*} + \frac{\partial\eta}{\partial t^*}\right) + \Delta\frac{\partial\eta}{\partial x^*} - 6\eta\frac{\partial\eta}{\partial x^*} - \frac{\partial^3\eta}{\partial x^{*3}} = \frac{\partial f\left(x^*\right)}{\partial x^*} \tag{7}$$

Superscript is omitted and equation (7) is re-written,

$$\frac{\partial\eta}{\partial t} + (\Delta - V)\frac{\partial\eta}{\partial x} - 6\eta\frac{\partial\eta}{\partial x} - \frac{\partial^3\eta}{\partial x^3} = \frac{\partial f\left(x\right)}{\partial x} \tag{8}$$

For simplicity, $\mathrm{K} = \Delta - V \approx 0$ is let to be too small.

For periodic function, a general form of sinusoidal term is used as

$$f(x) = A\ \sin(Bx) - d \tag{9}$$

Hence, equation (8) can re-written,

$$\frac{\partial \eta}{\partial t}-6\eta\frac{\partial \eta}{\partial x}-\frac{\partial^3 \eta}{\partial x^3}-\frac{\partial f(x)}{\partial x}=0. \qquad (10)$$

The simplified periodic forcing function KdV of equation (10) will be solved via HAM.

HAM and Forced KdV

Considering specific initial conditions, the rule of solution expression and the rule of solution existence in HAM [9], solution of forced KdV equation (10) is obtained through these steps,
Considering, nonlinear partial differential equation,

$$N\left[U\left(x,t\right)\right]=0 \qquad (11)$$

where N is a differential operator, $U\left(x,t\right)$ is unknown function , x and t is dependent variables.
Using zero-order deformation equation in HAM,

$$\left(1-q\right)\ell\left[U\left(x,t;q\right)-\eta_0\left(x,t\right)\right]=qc_0\mathrm{N}\left[U\left(x,t;q\right)\right] \qquad (12)$$

where c_0 is convergence parameter, ℓ as the auxiliary linear operator satisfying

$$\ell\left[d\right]=0 \qquad (13)$$

where d is constant.
Let, $q\in\left[0,1\right]$ is the embedding parameter,
in which it holds,

$$U\left(x,t;0\right)=\eta_0\left(x,t\right) \text{ and } U\left(x,t;1\right)=\eta\left(x,t\right). \qquad (14)$$

Using Taylor Series,

$$U\left(x,t;q\right)=\eta_0\left(x,t\right)+\sum_{m=1}^{\infty}\eta_m\left(x,t\right)q^m, \qquad (15)$$

where

$$\eta_m\left(x,t\right)=\frac{1}{m!}\frac{\partial^m U\left(x,t;q\right)}{\partial q^m}\bigg|_{q=0} \quad ; \quad m\geq 1. \qquad (16)$$

Since series convergent at $q=1$, equation (15) re-written

$$\eta\left(x,t\right)=\eta_0\left(x,t\right)+\sum_{m=1}^{\infty}\eta_m\left(x,t\right) \qquad (17)$$

Let, initial guess function [17],

$$\eta_0\left(x,t\right)=\frac{-2e^x}{\left(1+e^x\right)^2} \qquad (18)$$

And define the vectors,

$$\vec{\eta}_m\left(x,t\right)=\left\{\eta_0\left(x,t\right),\eta_1\left(x,t\right),\eta_2\left(x,t\right),........,\eta_m\left(x,t\right)\right\} \qquad (19)$$

Differentiating the zero-order deformation m-times with respect to the embedding parameter, $q=1$,

$$\ell\left[\eta_m(x,t)-\chi_m\eta_{m-1}(x,t)\right]=c_o R_m\left[\vec{\eta}_{m-1}(x,t)\right] \tag{20}$$

where

$$R_m\left[\vec{\eta}_{m-1}(x,t)\right]=\frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial q^{m-1}}\mathrm{N}\left[\sum_{n=0}\eta_n(x,t)q^n\right]\right\}\Bigg|_{q=0} \tag{21}$$

Hence, equation (10) will be employed in the HAM procedure as the follows

$$\mathrm{N}\left[\eta(x,t;q)\right]=\frac{\partial\eta(x,t;q)}{\partial t}-6\eta(x,t;q)\frac{\partial\eta(x,t;q)}{\partial x}-\frac{\partial^3\eta(x,t;q)}{\partial x^3}-\frac{\partial f(x)}{\partial x} \tag{22}$$

And equation (20) and (21) is rewritten

$$\ell\left[\eta_m(x,t)-\chi_m\eta_{m-1}(x,t)\right]=c_o\left[\frac{\partial\eta_{m-1}}{\partial t}-6\left(\sum_{i=0}^{m-1}\eta_i\frac{\partial\eta_{m-1-i}}{\partial x}\right)-\frac{\partial^3\eta_{m-1}}{\partial x^3}-\frac{1}{2}\frac{\partial f_{m-1}}{\partial x}\right] \tag{23}$$

with

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases} \tag{24}$$

And initial condition,

$$\eta_m(x,t;0)=0\ ,\ m\ge 1 \tag{25}$$

The *N*-th order approximation of $\eta(x,t)$ in HAM is given by

$$\eta(x,t)\approx\eta_0(x,t)+\sum_{m=1}^{N}\eta_m(x,t) \tag{26}$$

where, it is must be one of the solutions, if the HAM converges [9]. For simplicity, we let $A=1$, $B=1$ and *d* can be any constant value with condition it satisfies the constraints of depth in shallow water equation.

HAM methodology incorporating forcing KdV models from equations (11) till (26) are used to obtain analytic approximate solution of fKdV model.

HAM SOLUTION OF THE FKDV MODEL FLOW

HAM solution of fKdV model for the periodic plane found at seventh order approximation by using symbolic software. Using equation (26), HAM solution rewritten,

$$\eta(x,t)\approx\eta_0(x,t)+\sum_{m=1}^{7}\eta_m(x,t) \tag{27}$$

$$\eta(x,t)=\eta_0(x,t)+\eta_1(x,t)+\eta_2(x,t)+\eta_3(x,t)+\eta_4(x,t)+\eta_5(x,t)+\eta_6(x,t)+\eta_7(x,t) \tag{28}$$

HAM solution at 7th order approximation series shown in equation 29. The solution containing HAM convergence term, plotted at Figure 1 to identify the suitable convergence value. Due to limiting pages, solution shown partially.

$$
\begin{aligned}
\phi = & -\frac{2 e^{x}}{(1+e^{x})^{2}} + \frac{1}{720\,(1+e^{x})^{14}} \\
& \left(-1440\, e^{x} h\, t - 14\,400\, e^{2x} h\, t - 63\,360\, e^{3x} h\, t - 158\,400\, e^{4x} h\, t - 237\,600\, e^{5x} h\, t - \right. \\
& 190\,080\, e^{6x} h\, t + 190\,080\, e^{8x} h\, t + 237\,600\, e^{9x} h\, t + 158\,400\, e^{10x} h\, t + 63\,360\, e^{11x} h\, t + \\
& 14\,400\, e^{12x} h\, t + 1440\, e^{13x} h\, t - 7200\, e^{x} h^{2} t - 72\,000\, e^{2x} h^{2} t - 316\,800\, e^{3x} h^{2} t - \\
& 792\,000\, e^{4x} h^{2} t - 1\,188\,000\, e^{5x} h^{2} t - 950\,400\, e^{6x} h^{2} t + 950\,400\, e^{8x} h^{2} t + \\
& 1\,188\,000\, e^{9x} h^{2} t + 792\,000\, e^{10x} h^{2} t + 316\,800\, e^{11x} h^{2} t + 72\,000\, e^{12x} h^{2} t + \\
& 7200\, e^{13x} h^{2} t - 14\,400\, e^{x} h^{3} t - 144\,000\, e^{2x} h^{3} t - 633\,600\, e^{3x} h^{3} t - 1\,584\,000\, e^{4x} h^{3} t - \\
& 2\,376\,000\, e^{5x} h^{3} t - 1\,900\,800\, e^{6x} h^{3} t + 1\,900\,800\, e^{8x} h^{3} t + 2\,376\,000\, e^{9x} h^{3} t + \\
& 1\,584\,000\, e^{10x} h^{3} t + 633\,600\, e^{11x} h^{3} t + 144\,000\, e^{12x} h^{3} t + 14\,400\, e^{13x} h^{3} t - \\
& 14\,400\, e^{x} h^{4} t - 144\,000\, e^{2x} h^{4} t - 633\,600\, e^{3x} h^{4} t - 1\,584\,000\, e^{4x} h^{4} t - \quad \ldots\ldots\ldots \\
& 2\,376\,000\, e^{5x} h^{4} t - 1\,900\,800\, e^{6x} h^{4} t + 1\,900\,800\, e^{8x} h^{4} t + 2\,376\,000\, e^{9x} h^{4} t +
\end{aligned}
\tag{29}
$$

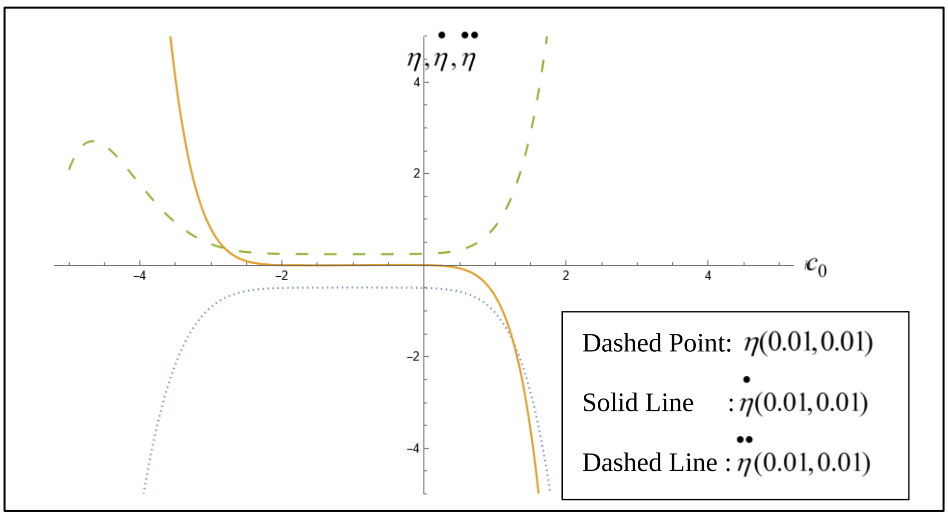
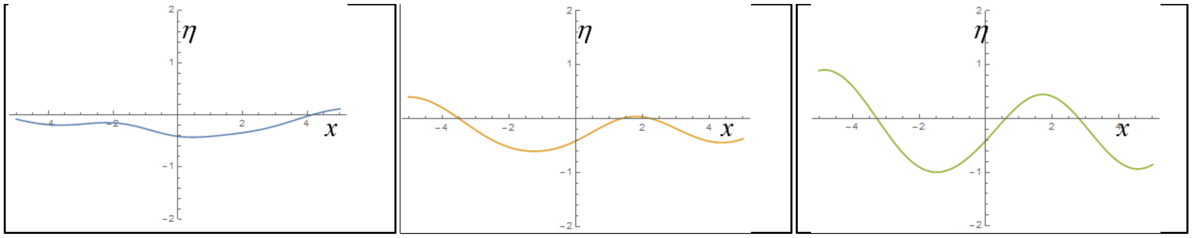


FIGURE 1. The c_0-curves according to 7th-order approximation for fKdV with bottom topography of equation (9).

Horizontal line segment in Figure 1 reveals the HAM solution is fully convergent and it confirms that solution can be found. Thus, the solution in the permissible region can be used to determine the interaction of waves correspond to periodic forcing function in fKdV equation. Based on the horizontal line segment, the permissible interval is $-2 < c_o < 0.25$, hereby HAM solution found at $c_0 = -1$.



(a) (b) (c)

FIGURE 2. HAM solution at (a) t =0.5, (b) t=1.0 and (c) t=1.5.

Figure 2 depicts HAM solution of periodic forcing fKdV equation at a particular convergence value, c_0 = -1. The solution had been shown for various t. The Figure 2(a) plotted at t = 0.5, resembles the initial disturbance of wave profile. The initial guess function given in equation (18), is a solution for KdV equation in which it has good balancing effect of nonlinearity and dispersion. Hereby, in figure 2(a), nonlinearity seems weak, and much elevation not seen. Figure 2(b) showed HAM solution at t=1 in which wave profile is elevated and a smooth flow is observed. In the Figure 2 (c), much elevated wave profile is observed.

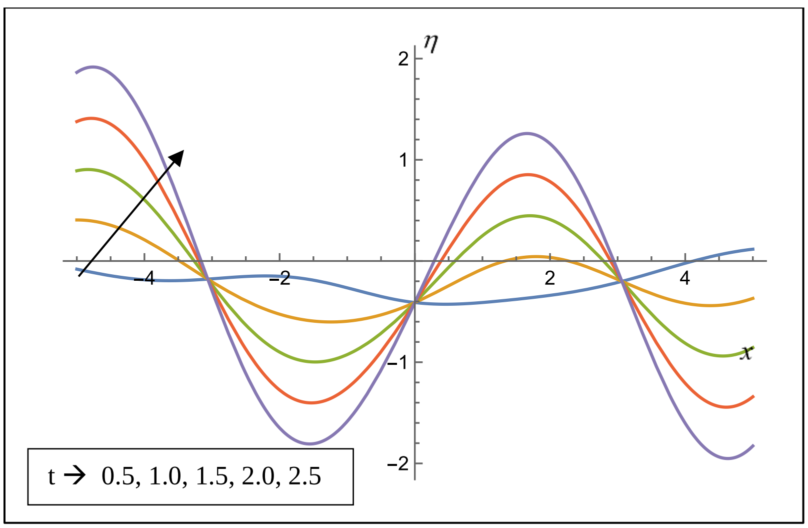


FIGURE 3. Convergent HAM solution for $0 < t < 2.5$.

Figure 3 showed wave profile for t in between 0 and 2.5. The figure 3 confirm that the amplitude of waves increases correspondingly with increment of *t*-values. However, the x-position of crest and trough is maintained. Hereby, it can be seen, HAM solution had exhibits a cyclic behaviour. It is just the wave profile moved from initial position (t=0) after t increases.

Conclusion

In this paper, a non-homogeneous forced Korteweg-de Vries equation had been approximated using HAM method. The interaction between solitary waves and periodic forcing function is investigated. The forcing function is simplified to provide a general solution of fKdV with periodic function. Horizontal Line Segment validate that HAM solution is fully convergent and it shows HAM solution can be used to analyze the impact of periodic forcing function onto solitary waves. It is found the solitary waves disturbed by presence of forcing functions. The wave profile seems moved forward and backward from initial position. However, it is found the wave profile maintain steady and does not loose its balancing effect. The amplitude of the wave seems increasing, but the x-position of crest and trough seems to be same. The wavelength of wave profile is stable and probably its due to fix periodic forcing function.

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